

Cosmic strings in $f(R, L_m)$ gravity

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We consider Kasner-type static, cylindrically symmetric interior string solutions in the $f(R, L_m)$ theory of modified gravity. The physical properties of the string are described by an anisotropic energy-momentum tensor satisfying the condition $T_t^t = T_z^z$; that is, the energy density of the string along the z -axis is equal to minus the string tension. As a first step in our study we obtain the gravitational field equations in the $f(R, L_m)$ theory for a general static, cylindrically symmetric metric, and then for a Kasner-type metric, in which the metric tensor components have a power law dependence on the radial coordinate r . String solutions in two particular modified gravity models are investigated in detail. The first is the so-called “exponential” modified gravity, in which the gravitational action is proportional to the exponential of the sum of the Ricci scalar and matter Lagrangian, and the second is the “self-consistent model”, obtained by explicitly determining the gravitational action from the field equations under the assumption of a power law dependent matter Lagrangian. In each case, the thermodynamic parameters of the string, as well as the precise form of the matter Lagrangian, are explicitly obtained.

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I. INTRODUCTION

The recently released Planck satellite data of the 2.7 K Cosmic Microwave Background (CMB) full sky survey [1] have generally confirmed the standard Λ Cold Dark Matter (Λ CDM) cosmological model. Nonetheless, observations of high redshift type Ia supernovae have convincingly shown that our universe is presently undergoing a period of accelerated expansion, for which the most natural explanation would be the presence of some form of dark energy, with an equation of state parameter $w = -1.018 \pm 0.057$ for a flat universe [2]. From a theoretical point of view, the necessity of explaining dark energy, as well as the second dominant component of the universe, dark matter [3], raises the fundamental question of whether the standard Einstein–Hilbert action ($S = \int (R/2 + L_m) \sqrt{-g} d^4x$, where R is the scalar curvature, and L_m is the matter Lagrangian density), in which matter is minimally coupled to geometry, can give an appropriate quantitative description of the universe on all scales, from the boundary of the solar system to the edge of the horizon.

A gravitational theory with an explicit coupling between an arbitrary function of the scalar curvature and the Lagrangian density of matter was proposed in [4]. The gravitational action of this theory is of the form $S = \int \{f_1(R) + [1 + \lambda f_2(R)] L_m\} \sqrt{-g} d^4x$, where λ is a constant, and $f_1(R)$ and $f_2(R)$ are arbitrary functions of the Ricci scalar. In this model an extra force acting on massive test particles arises, and the mo-

tion is no longer geodesic. Moreover, in this framework, one can also explain dark matter [4]. The initial linear geometry–matter coupling introduced in [4] was extended in [5], and a maximal extension of the Einstein–Hilbert action with geometry–matter coupling, of the form $S = \int d^4x \sqrt{-g} f(R, L_m)$, was considered in [6]. Geometry–matter couplings in the presence of scalar fields were discussed in [7] and the cosmological and astrophysical implications of the $f(R, L_m)$ type gravity theories were investigated in [8]. For a recent review of the $f(R, L_m)$ model see [9].

Topological defects, including strings, are expected to have formed via the Kibble mechanism [10] during in phase transitions in the early Universe [11–13]. In particular, magnetic monopoles are generic features of Grand Unified Theories (GUTs) [14, 15], with expected mass scales in the region of $m \sim m_{GUT}/\alpha_{GUT} \sim 10^{17}$ GeV, where $m_{GUT} \sim 10^{15}$ GeV is the expected grand unification scale and α_{GUT} is the coupling constant. Though not completely ruled out by existing data, the mass and number densities of monopoles and domain walls, resulting from the spontaneous breaking of spherical and parity symmetry, respectively, are tightly constrained by observations [13, 16–22].

For the time being, however, we limit ourselves to consideration of the most viable species of topological defects which may exist as remnants from the early universe; cosmic strings. In field theory models, strings form whenever an axial symmetry is spontaneously broken. They are either “infinitely” long (that is, spanning the horizon), or exist in the form of loops. As shown by both analytical studies [23] and numerical simulations [24], cosmic string networks typically reach a scaling solution in which their contribution to the total energy density of the universe

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becomes constant. This is due to string self-intersection and the resulting “chopping off” of loops from the long string network, which then subsequently decay via gravitational radiation [25] or gauge particle emission [26].

Remarkably, this behavior also seems to occur in models motivated by string theory, in which cosmic F -strings (fundamental, Nambu–Goto strings [27] of macroscopic size) and D -strings (one-dimensional D -branes [28]) or, more generally, bound states of p F -strings and q D -strings called (p, q) -strings, exist in space-times with compact extra dimensions [29]. In such a scenario, one may expect the inter-commuting probability of the string network to be significantly less than unity, leading to reduced loop formation [30] but, to date, predictions of scaling behavior based on analytic and numerical studies appear to be robust [31].

Though there are a great many field-theoretic and string theory inspired models of cosmic string formation, evolution and decay within the existing literature, at present, almost all studies of the gravitational properties of strings have been conducted in the context of general relativity or, at least, in models for which non-standard gravitational effects (such as those caused by the presence of compact extra dimensions) may be neglected. Notable exceptions include a handful of studies in $f(R)$ gravity [32, 33, 35], teleparallel theories [36], brane worlds [37], Kaluza–Klein models [38], Lovelock [39], Gauss–Bonnet [40], Born–Infeld [41] and bimetric [42] gravity theories, and a few other, less mainstream, alternatives [43], while more comprehensive bodies of work exists for strings in scalar-tensor theories [44], including string theory-inspired dilation gravity [45], and gravitational theories with torsion [46]. The possible existence of Bose-Einstein condensate strings was investigated in [47].

It is the goal of the present paper to consider string-type solutions in the $f(R, L_m)$ gravity theory and, to the best of the author’s knowledge, this study constitutes the first analysis of such solutions in a gravitational theory with arbitrary nonminimal matter-geometry coupling.

Of the string-type solutions obtained in $(3 + 1)$ -dimensional modified gravity theories, those found in general $f(R)$ gravity [32–34] bear closest resemblance to the solutions considered in the present work. In [32], Azadi *et al* considered cylindrically symmetric solutions with constant Ricci curvature. They found the unique solution for $R = 0$, which is also recovered in Sects. V–VI as a specific case, corresponding to cylindrically symmetric metrics obeying both sets of Kasner conditions [48, 49]. In addition, they found two further families of vacuum solutions for which $R = \text{const.} \neq 0$, in which the Ricci scalar was found to play the role of a cosmological constant term, giving rise to an $f(R)$ gravity analogue of the Linet–Tian solution [50], for cosmic strings in an Einstein gravity space-time with $\Lambda \neq 0$. In [33], Momeni and Gholizade showed that the solution previously obtained in [32] is, in fact, only one member of the most generalized Tian family in general relativity and, as such,

is physically applicable to the exterior of a cosmic string.

However, the results derived in the present study seem, at first, to contrast with those presented in [32, 33], since we determine the Ricci scalar for general, Kasner-type space-times in $f(R, L_m)$ gravity to be of the form $R = R_0(a, b, c)/r^2$, where $R_0(a, b, c) = 0$ when both sets of Kasner conditions are satisfied. Since the $f(R, L_m)$ theory naturally encompasses all cases in which the gravitational Lagrangian is of the form $f(R) + L_m$, it necessarily implies that cylindrically symmetric vacuum solutions in the $f(R)$ theory, for which $R = \text{const.} \neq 0$, cannot be of Kasner form. By the results presented in [33], the physical implication of this statement is that Linet–Tian type solutions in $f(R)$ gravity, in which the cosmological constant term in the field equations is “generated” by the constant Ricci scalar, *cannot* be described by the Kasner metric, and more general cylindrically symmetric metrics must be considered in order to obtain them. This contrasts with the case in general relativity, in which a “Kasner-type” solution, corresponding to $R = 4\Lambda = \text{const.}$ exists [51], but is not of the standard Kasner form (i.e obeying both sets of Kasner conditions, for which the unique vacuum solution to Einstein’s field equations with $\Lambda = 0$ is recovered).

Some additional, nonvacuum, solutions in $f(R)$ gravity were also found in [34], using a self-consistent method in which the form of $f(R)$ was determined from the general field equations using a cylindrically symmetric metric. Though not generally of Kasner form, a more careful comparison of these solutions with the nonvacuum solutions obtained in the present work warrants further study, but is beyond the cope of this paper.

For vacuum strings in the wire approximation [52], or genuine F -strings governed by the Nambu–Goto action [27], in which the string world-sheet carries no additional currents or fluxes, the condition $\mathcal{T} = -\tilde{m}$, where \tilde{m} is the inertial mass per unit length and \mathcal{T} is the tension, holds for a static, straight string (commonly assumed to lie parallel to the z -axis). This condition arises from general Lorentz invariance and, specifically, from boost invariance along the string length. For vacuum strings of finite width and, potentially, nonuniform mass densities in the radial direction, r (such as the Abelian-Higgs string [53]), the equivalent condition is $T_t^t = T_z^z$, or $\rho(r) = -p_z(r)$, where $\rho(r)$ and $p_z(r)$ denote the mass density and thermodynamic pressure per unit volume within the string core. Since, for a cylindrically symmetric distribution of matter, these quantities depend only on the radial coordinate, integration over r yields the effective mass per unit length, $\tilde{m} = \int T^{tt} \sqrt{-g} dr$, and tension $\mathcal{T} = \int T^{zz} \sqrt{-g} dr$, of the string-type field configuration, which thus also satisfy $\mathcal{T} = -\tilde{m}$ [54]. For the sake of simplicity, we restrict our attention to vacuum strings throughout the following analysis and impose the condition $T_t^t = T_z^z$ on the matter sector.

The structure of this paper is then as follows. In Sect. II, we give a brief outline of the $f(R, L_m)$ gravity theory, and we derive the general field equations and con-

ervation equations from the action. The form of these equations is determined for a general static, cylindrically symmetric source in Sect. III A. Specific Kasner type solutions [48, 49] of the field equations are presented in Sect. III B. In Sect. IV, a specific choice is made for the function f , and the exponential modified gravity theory, $f(R, L_m) = \Lambda \exp[(1/2\Lambda)R + (1/\Lambda)L_m]$, which reduces naturally to Einstein gravity in the low energy limit, is chosen as a toy model. Solutions with self-consistent matter Lagrangians are derived in Sect. V. That is, the self-consistent form of $f(R, L_m)$ is determined by imposing the condition $T_t^t = T_z^z$ on the matter sector, which may be seen as a physical requirement corresponding to the existence of vacuum/Nambu–Goto strings, and by invoking the correspondence principle, which states that the standard general relativistic limit of the modified gravity action (i.e. the Einstein–Hilbert action), must exist for appropriate choices of the arbitrary parameters and functions of the model. Specific, Kasner-type solutions with self-consistent matter Lagrangians are investigated in Sect. VI, for a simple choice of matter Lagrangian dependence in f , $f(R, L_m) \propto L_m^q$, $q = \text{constant} \neq 1$, and two distinct string-type solutions are found to exist. These correspond to two self-consistent sets of constraints, which may be imposed on the free parameters of the Kasner metric. Section VII contains a brief summary of the conclusions and discussion of the main results of this analysis. In the present paper we use the natural system of units, with $8\pi\mathcal{G} = c_{\text{light}} = 1$, where c_{light} denotes the speed of light, and \mathcal{G} the gravitational constant, respectively.

II. $f(R, L_m)$ GRAVITY

In this section, we first briefly summarize the $f(R, L_m)$ theory of gravity and introduce the gravitational action, the field equations and the conservation equations, respectively. The $f(R, L_m)$ theory is based on the assumption that the gravitational Lagrangian is given by an arbitrary function f of the Ricci scalar R and of the matter Lagrangian L_m [6]. This theory represents a maximal extension of the Hilbert–Einstein action, and the action takes the form [6]

$$S = \int f(R, L_m) \sqrt{-g} d^4x, \quad (1)$$

where $\sqrt{-g}$ is the square root of the determinant of the metric tensor. We require that the function f be analytic in both R and L_m ; that is, f must possess a Taylor series expansion about any point (R, L_m) . The energy-momentum tensor of the matter is defined by [55]

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \left[\frac{\partial(\sqrt{-g}L_m)}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\lambda} \frac{\partial(\sqrt{-g}L_m)}{\partial(g^{\mu\nu}/\partial x^\lambda)} \right]. \quad (2)$$

Thus, by assuming that the Lagrangian density of the matter, L_m , depends only on the metric tensor $g_{\mu\nu}$,

and not on its derivatives, we obtain $T_{\mu\nu} = g_{\mu\nu}L_m - 2\partial L_m/\partial g^{\mu\nu}$.

By varying the action with respect to the metric we obtain the following field equations for the $f(R, L_m)$ gravity theory [6]:

$$f_R(R, L_m) R_{\mu\nu} + (g_{\mu\nu}\square - \nabla_\mu \nabla_\nu) f_R(R, L_m) - \frac{1}{2} \left[f(R, L_m) - f_{L_m}(R, L_m) L_m \right] g_{\mu\nu} = \frac{1}{2} f_{L_m}(R, L_m) T_{\mu\nu}, \quad (3)$$

where we have denoted $f_R(R, L_m) = \partial f(R, L_m)/\partial R$ and $f_{L_m}(R, L_m) = \partial f(R, L_m)/\partial L_m$, respectively. Alternatively, Eq. (3) can be written as

$$R_{\mu\nu} = \frac{1}{f_R(R, L_m)} \hat{P}_{\mu\nu} f_R(R, L_m) + \Lambda_{eff}(R, L_m) g_{\mu\nu} + G_{eff}(R, L_m) T_{\mu\nu}, \quad (4)$$

where we have denoted

$$\hat{P}_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu} \square, \quad (5)$$

$$\Lambda_{eff}(R, L_m) = \frac{1}{2f_R(R, L_m)} \left[f(R, L_m) - f_{L_m}(R, L_m) L_m \right], \quad (6)$$

and

$$G_{eff}(R, L_m) = \frac{1}{2} \frac{f_{L_m}(R, L_m)}{f_R(R, L_m)}. \quad (7)$$

The differential operators $\nabla_\mu \nabla_\nu$ and \square are given by

$$\nabla_\mu \nabla_\nu F(x^\lambda) = \frac{\partial^2 F}{\partial x^\mu \partial x^\nu} - \Gamma_{\mu\nu}^\alpha \frac{\partial F}{\partial x^\alpha}, \quad (8)$$

and

$$\square F(x^\lambda) = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial F}{\partial x^\nu} \right), \quad (9)$$

where $F(x^\lambda)$ is an arbitrary function of the coordinates x^λ , and the $\Gamma_{\mu\nu}^\alpha$, defined as

$$\Gamma_{\alpha\mu\nu} = \frac{1}{2} \left(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha} \right), \quad (10)$$

are the Christoffel symbols associated to the metric.

With the choice $f(R, L_m) = R/2 + L_m$ (the Einstein–Hilbert Lagrangian), we recover the standard Einstein field equations of general relativity, $R_{\mu\nu} - (1/2)g_{\mu\nu}R = T_{\mu\nu}$. By choosing $f(R, L_m) = f_1(R) + f_2(R)G(L_m)$, where f_1 and f_2 are arbitrary functions of the Ricci scalar, and G is a function of the matter Lagrangian density, we may reobtain the field equations of the modified gravity with arbitrary curvature-matter coupling, introduced in [4], for an appropriate choice of G .

The contraction of Eq. (3) gives the following relation between the Ricci scalar R , the matter Lagrangian density L_m , and the trace of the energy-momentum tensor, $T = T^\mu_\mu$,

$$R + \frac{3}{f_R(R, L_m)} \square f_R(R, L_m) - 4\Lambda(R, L_m) = G_{eff}(R, L_m) T. \quad (11)$$

By eliminating the $\square f_R(R, L_m)$ term using Eqs. (3) and (11), we obtain the gravitational field equations in an alternative form as

$$R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} + \frac{1}{3} \Lambda(R, L_m) g_{\mu\nu} = G_{eff}(R, L_m) \times \left(T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu} \right) + \frac{1}{f_R(R, L_m)} \nabla_\mu \nabla_\nu f_R(R, L_m). \quad (12)$$

Taking the covariant divergence of Eq. (3), with the use of the mathematical identity [56],

$$\square \nabla_\nu - \nabla_\nu \square = R_{\mu\nu} \nabla^\mu, \quad (13)$$

then gives the following equation for the divergence of the energy-momentum tensor $T_{\mu\nu}$,

$$\begin{aligned} \nabla^\mu T_{\mu\nu} &= \nabla^\mu \ln[f_{L_m}(R, L_m)] \{L_m g_{\mu\nu} - T_{\mu\nu}\} \\ &= 2 \nabla^\mu \ln[f_{L_m}(R, L_m)] \frac{\partial L_m}{\partial g^{\mu\nu}}. \end{aligned} \quad (14)$$

The explicit derivation of Eq. (14) is also presented in Appendix A. The requirement that the energy-momentum tensor for matter be conserved, $\nabla^\mu T_{\mu\nu} = 0$, then yields an effective functional relation between the matter Lagrangian density and the total Lagrangian $f(R, L_m)$,

$$\nabla^\mu \ln[f_{L_m}(R, L_m)] \frac{\partial L_m}{\partial g^{\mu\nu}} = 0. \quad (15)$$

Thus, once the matter Lagrangian density is known, by an appropriate choice of $f(R, L_m)$, one can construct, at least in principle, conservative gravity models with arbitrary curvature-matter coupling.

III. FIELD EQUATIONS FOR $f(R, L_m)$ MODIFIED GRAVITY ASSUMING A STATIC CYLINDRICAL SYMMETRIC SOURCE

In the present section we consider the gravitational field equations describing explicit string solutions in the $f(R, L_m)$ gravity theory. We assume a cylindrically symmetric geometry in which the source term is represented in the form of an anisotropic fluid with three distinct pressures components along the r -, ϕ - and z -axes (in cylindrical polar coordinates, (t, r, ϕ, z)). The simplest model for a cosmic string is the so-called ‘‘gauge string’’ [57], which is an idealized cylindrical mass distribution with a finite radial extension having $T_t^t = T_z^z$ as the only nonvanishing components of the energy-momentum tensor. In the following we restrict our analysis to a similar configuration, but with a more general pressure distribution, for which $T_r^r \neq 0$ and $T_\phi^\phi \neq 0$, respectively.

A. String geometry and field equations in $f(R, L_m)$ gravity

We assume a cylindrical symmetry metric, giving a line element of the form [58]

$$ds^2 = N^2(r) dt^2 - dr^2 - L^2(r) d\phi^2 - K^2(r) dz^2, \quad (16)$$

where $N(r)$, $L(r)$ and $K(r)$ are arbitrary functions of the radial coordinate r . The nonzero Christoffel symbols associated to the metric (16) are given by

$$\begin{aligned} \Gamma_{10}^0 &= \frac{N'(r)}{N(r)}, \Gamma_{00}^1 = N(r)N'(r), \Gamma_{12}^2 = \frac{L'(r)}{L(r)}, \\ \Gamma_{22}^1 &= -L(r)L'(r), \Gamma_{13}^3 = \frac{K'(r)}{K(r)}, \Gamma_{33}^1 = -K(r)K'(r), \end{aligned} \quad (17)$$

where a prime denotes the derivative with respect to r . The operator \square is given by

$$\square = -\frac{1}{N(r)L(r)K(r)} \frac{d}{dr} \left[N(r)L(r)K(r) \frac{d}{dr} \right], \quad (18)$$

while for the covariant derivatives of an arbitrary function $f(r)$ we obtain

$$\begin{aligned} \nabla_t \nabla_t f &= -N(r)N'(r) \frac{df}{dr}, \nabla_r \nabla_r f = \frac{d^2 f}{dr^2}, \\ \nabla_\phi \nabla_\phi f &= L(r)L'(r) \frac{df}{dr}, \nabla_z \nabla_z f = K(r)K'(r) \frac{df}{dr}. \end{aligned}$$

With the line element (16), the components of the Ricci tensor are [58]

$$\begin{aligned} R_t^t &= \frac{(LKN')'}{NLK}, R_r^r = \frac{N''}{N} + \frac{L''}{L} + \frac{K''}{K}, \\ R_\phi^\phi &= \frac{(NKL')'}{NLK}, R_z^z = \frac{(NLK')'}{NLK}. \end{aligned} \quad (19)$$

The source term in the field equations is given by the energy-momentum tensor with the following components [58]:

$$\begin{aligned} T_t^t &= \rho(r), T_r^r = -p_r(r), T_\phi^\phi = -p_\phi(r), \\ T_z^z &= -p_z(r) = \rho(r). \end{aligned} \quad (20)$$

Therefore the field equations describing cylindrically symmetric string-type solutions in $f(R, L_m)$ gravity can be written as

$$\begin{aligned} \frac{(LKN')'}{NLK} &= G_{eff}(R, L_m) \rho + \Lambda(R, L_m) - \frac{1}{f_R(R, L_m)} \\ &\times \left[\frac{1}{NLK} \frac{d}{dr} \left(NLK \frac{d}{dr} \right) + \frac{N'}{N} \frac{d}{dr} \right] f_R(R, L_m), \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{N''}{N} + \frac{L''}{L} + \frac{K''}{K} &= -G_{eff}(R, L_m) p_r + \Lambda(R, L_m) \\ &+ \frac{1}{f_R(R, L_m)} \left[\frac{1}{NLK} \frac{d}{dr} \left(NLK \frac{d}{dr} \right) - \frac{d^2}{dr^2} \right] f_R(R, L_m), \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{(NKL')'}{NLK} &= -G_{eff}(R, L_m) p_\phi + \Lambda(R, L_m) \\ &+ \frac{1}{f_R(R, L_m)} \left[\frac{1}{NLK} \frac{d}{dr} \left(NLK \frac{d}{dr} \right) - \frac{L'}{L} \frac{d}{dr} \right] \\ &\times f_R(R, L_m), \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{(NLK')'}{NLK} &= G_{eff}(R, L_m) \rho + \Lambda(R, L_m) \\ &+ \frac{1}{f_R(R, L_m)} \left[\frac{1}{NLK} \frac{d}{dr} \left(NLK \frac{d}{dr} \right) - \frac{K'}{K} \frac{d}{dr} \right] \\ &\times f_R(R, L_m). \end{aligned} \quad (24)$$

Generally, the regularity of the geometry on the symmetry axis is imposed via the initial conditions,

$$L(0) = 0, L'(0) = 1, N(0) = 1, N'(0) = 0. \quad (25)$$

We define the radius R_s of the string as the radius for which the radial pressure $p_r(r)$ vanishes, $p_r(R_s) = 0$. Additionally, we may also impose the conditions $p_\phi(R_s) = 0$, and $\int_{r=0}^{R_s} p_r(r) dr = 0$, and $\int_{r=0}^{R_s} p_\phi(r) dr = 0$, respectively. The physical reasoning behind these conditions is that, if they did not hold, the string core would either be expanding or contracting. However, for some string models these conditions may not hold, thus allowing the possibility of a nonzero surface energy density and tangential pressure.

Next we consider the relations that follows from the “conservation” of the energy-momentum tensor, given by Eq. (14). Since all the components of the energy-momentum tensor and of the function $f(R, L_m)$ are independent of the coordinates (t, ϕ, z) , we have

$$\nabla_\mu T_t^\mu = 0, \nabla_\mu T_\phi^\mu = 0, \nabla_\mu T_z^\mu = 0. \quad (26)$$

Furthermore, the divergence of the T_ν^μ can be obtained in a general form as [55]

$$\nabla_\mu T_\nu^\mu = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} T_\nu^\mu) - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\nu} T^{\alpha\beta}. \quad (27)$$

It follows that Eq. (26) is identically satisfied, with the only potentially nonzero component of the divergence being given by

$$\nabla_\mu T_r^\mu = \frac{1}{\sqrt{-g}} \frac{d}{dr} (\sqrt{-g} T_r^r) - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial r} T^{\alpha\beta}. \quad (28)$$

With the use of Eq. (28), the “conservation equation” for the cosmic string in the $f(R, L_m)$ gravity theory takes the form

$$\begin{aligned} -\frac{dp_r}{dr} - (\rho + p_r) \frac{N'}{N} + (p_\phi - p_r) \frac{L'}{L} + (p_z - p_r) \frac{K'}{K} = \\ \frac{d}{dr} \ln f_{L_m}(R, L_m) (L_m + p_r). \end{aligned} \quad (29)$$

and, since $p_z = -\rho$, this is equivalent to

$$\begin{aligned} -\frac{dp_r}{dr} - (\rho + p_r) \left(\frac{N'}{N} + \frac{K'}{K} \right) + (p_\phi - p_r) \frac{L'}{L} = \\ \frac{d}{dr} \ln f_{L_m}(R, L_m) (L_m + p_r). \end{aligned} \quad (30)$$

To obtain a description of the physical properties of the string we introduce the Tolman mass per unit length $M(r)$ of this system, defined as [59]

$$\begin{aligned} M(r) &= 2\pi \int_0^r (\rho + p_r + p_\phi + p_z) N^2 L dr = \\ &2\pi \int_0^r (p_r + p_\phi) N^2 L dr, \end{aligned} \quad (31)$$

along with the parameter

$$\begin{aligned} W(r) &= -2\pi \int_0^r (\rho - p_r + p_\phi - p_z) N^2 L dr = \\ &-2\pi \int_0^\infty (2\rho - p_r + p_\phi) N^2 L dr, \end{aligned} \quad (32)$$

which can be related to the angular deficit of the space-time via the “angular” Einstein equation [59]. With the use of the field equations Eqs. (21)–(24) we obtain

$$\begin{aligned} M(r) &= 2\pi \int_0^r \frac{1}{G_{eff}(R, L_m)} \left\{ 2\Lambda_{eff}(R, L_m) + \frac{1}{f_R(R, L_m)} \left[\frac{2}{NLK} \frac{d}{dr} \left(NLK \frac{d}{dr} \right) - \frac{d^2}{dr^2} - \frac{L'}{L} \frac{d}{dr} \right] f_R(R, L_m) \right\} N^2 L dr - \\ &2\pi \int_0^\infty \frac{1}{G_{eff}(R, L_m)} \left[\frac{N''}{N} + \frac{L''}{L} + \frac{K''}{K} + \frac{(NKL')'}{NLK} \right] N^2 L dr, \end{aligned} \quad (33)$$

and

$$\begin{aligned} W(r) &= -2\pi \int_0^r \frac{1}{G_{eff}(R, L_m)} \left[\frac{(LKN')'}{NLK} + \frac{N''}{N} + \frac{L''}{L} + \frac{K''}{K} - \frac{(NKL')'}{NLK} + \frac{(NLK')'}{NLK} \right] N^2 L dr - \\ &2\pi \int_0^\infty \frac{1}{G_{eff}(R, L_m)} \left\{ 2\Lambda_{eff}(R, L_m) - \frac{1}{f_R(R, L_m)} \left[\frac{d^2}{dr^2} + \left(\frac{N'}{N} - \frac{L'}{L} + \frac{K'}{K} \right) \frac{d}{dr} \right] f_R(R, L_m) \right\} N^2 L dr, \end{aligned} \quad (34)$$

respectively. In the following we denote the total value

of the Tolman mass and the value of the angular deficit

parameter on the string surface by $M = M(R_s)$ and $W = W(R_s)$, respectively.

B. The string gravitational field equations for the Kasner metric

We now investigate the possibility of the existence of Kasner-type solutions for static, cylindrically symmetric strings in $f(R, L_m)$ gravity. In standard general relativity, by neglecting the effect of the cosmological constant, the unique, static, cylindrically symmetric vacuum solution of the Einstein field equations is given by the Kasner metric [48, 49],

$$ds^2 = (kr)^{2a} dt^2 - dr^2 - \beta^2 (kr)^{2(b-1)} r^2 d\phi^2 - (kr)^{2c} dz^2, \quad (35)$$

where k sets the length scale and β is a constant, related to the deficit angle of the conical space-time. The Kasner metric is characterized by two free parameters which, for the unique vacuum solution, satisfy the Kasner conditions [48, 49],

$$a + b + c = a^2 + b^2 + c^2 = 1. \quad (36)$$

The derivation of the Kasner metric in standard general relativity is presented in Appendix B. The physical interpretation of the free parameters a, b, c of the Kasner metric is of fundamental importance in the context of the description of cosmic strings, as is the relation between these parameters and the internal properties of the matter distribution. The simplest model for a cosmic string, the gauge string [57], belongs to a very simple class of Kasner-type solutions, with $a = c = 0, b = 1$, which is locally flat. However, as one can see from Eq. (35), in the gauge string model, there is an important global non-trivial effect, namely, that the string geometry is that of a cone. From a geometrical point of view the parameter β in the Kasner solution is directly related to the conic angular deficit [60, 61], which, in turn, is also related to the mass distribution of the source [62–66].

In the case of a gauge string (in general relativity), there is an explicit relation between the angular deficit $\Delta\phi = 2\pi(1 - \beta)$ and the inertial mass (per unit length) \tilde{m} , given by $\Delta\phi = 8\pi G\tilde{m}$. This relation was first obtained by using a linear approximation for the case of an infinitesimally thin source [67]. The same relation was obtained by solving the full nonlinear Einstein equations around a uniform source ($T_t^t = \text{constant}$) with a finite

radius [68, 69], and in the case of a nonuniform source [70]. Since the space-time around a gauge string is locally flat, the angular deficit is the only geometrical (and physical) evidence of its existence, thus allowing the possibility of observationally testing the presence of cosmic strings. The next step in the study of gravitating cosmic strings requires the analysis of the more realistic, Abelian-Higgs type models. In this context the analysis of the full coupled field equations for the gravitational field and matter fields was performed in [71].

Hence, for a Kasner-type metric, the gravitational field equations for the cylindrically symmetric static string in $f(R, L_m)$ gravity (21)–(24) take the form

$$\frac{(a+b+c-1)a}{r^2} = G_{eff}(R, L_m)\rho + \Lambda_{eff}(R, L_m) - \frac{1}{f_R(R, L_m)} \left[\frac{d^2}{dr^2} + \frac{2a+b+c}{r} \frac{d}{dr} \right] f_R(R, L_m), \quad (37)$$

$$\frac{a^2 + b^2 + c^2 - (a+b+c)}{r^2} = -G_{eff}(R, L_m)p_r + \Lambda_{eff}(R, L_m) + \frac{a+b+c}{r} \frac{d}{dr} \ln f_R(R, L_m), \quad (38)$$

$$\frac{(a+b+c-1)b}{r^2} = -G_{eff}(R, L_m)p_\phi + \Lambda(R, L_m) + \frac{1}{f_R(R, L_m)} \left[\frac{d^2}{dr^2} + \frac{a+c}{r} \frac{d}{dr} \right] f_R(R, L_m), \quad (39)$$

$$\frac{(a+b+c-1)c}{r^2} = G_{eff}(R, L_m)\rho + \Lambda_{eff}(R, L_m) + \frac{1}{f_R(R, L_m)} \left[\frac{d^2}{dr^2} + \frac{a+b}{r} \frac{d}{dr} \right] f_R(R, L_m). \quad (40)$$

The conservation equation (30) becomes

$$-\frac{dp_r}{dr} - (\rho + p_r) \frac{a+c}{r} + (p_\phi - p_r) \frac{b}{r} = \frac{d}{dr} \ln f_{L_m}(R, L_m) (L_m + p_r). \quad (41)$$

and the Ricci scalar is given by $R = R_0/r^2$, where

$$R_0 = a^2 + b^2 + c^2 + (a+b+c)(a+b+c-2). \quad (42)$$

The Tolman mass per unit length $M(r)$ and the $W(r)$ parameter, which controls the angular deficit of the string in the Kasner metric, are obtained as

$$M(r) = 2\pi\beta k \int_0^r \frac{(1-b)(a+b+c) + b - (a^2 + b^2 + c^2)}{G_{eff}(R, L_m)} (kr)^{2a+b-2} dr + \frac{2\pi\beta}{k} \int_0^\infty \frac{(kr)^{2a+b}}{G_{eff}(R, L_m)} \left\{ 2\Lambda_{eff}(R, L_m) + \frac{1}{f_R(R, L_m)} \left[\frac{d^2}{dr^2} + \frac{2a+b+2c}{r} \frac{d}{dr} \right] f_R(R, L_m) \right\} dr \quad (43)$$

and

$$W(r) = -2\pi k\beta \int_0^r \frac{1}{G_{eff}(R, L_m)} [(a+b+c-1)(a-b+c-1) + a^2 + b^2 + c^2 - 1] (kr)^{2a+b-2} dr - \frac{2\pi\beta}{k} \int_0^\infty \frac{(kr)^{2a+b}}{G_{eff}(R, L_m)} \left\{ 2\Lambda_{eff}(R, L_m) - \frac{1}{f_R(R, L_m)} \left[\frac{d^2}{dr^2} + \frac{a-b+c}{r} \frac{d}{dr} \right] f_R(R, L_m) \right\} dr, \quad (44)$$

respectively.

grangian density of the form [6]

$$f(R, L_m) = \Lambda \exp\left(\frac{1}{2\Lambda}R + \frac{1}{\Lambda}L_m\right), \quad (45)$$

where $\Lambda > 0$ is an arbitrary constant. In the limit $(1/2\Lambda)R + (1/\Lambda)L_m \ll 1$, we obtain

$$f(R, L_m) \approx \Lambda + \frac{R}{2} + L_m + \dots \quad (46)$$

That is, we recover the full Einstein–Hilbert gravitational Lagrangian with a cosmological constant.

With this choice of Lagrangian density the gravitational field equations take the form

IV. KASNER-TYPE STRING SOLUTIONS WITH A GIVEN FORM OF THE LAGRANGIAN DENSITY: EXPONENTIAL $f(R, L_m)$ GRAVITY

As a first example of string-type cylindrically symmetric solutions in $f(R, L_m)$ gravity we consider a La-

$$R_{\mu\nu} = (\Lambda - L_m)g_{\mu\nu} + T_{\mu\nu} - \frac{1}{\Lambda} \left[\left(\frac{1}{2} \nabla_\mu \nabla^\mu R + \nabla_\mu \nabla^\mu L_m \right) g_{\mu\nu} - \left(\frac{1}{2} \nabla_\mu \nabla_\nu R + \nabla_\mu \nabla_\nu L_m \right) \right] - \frac{1}{\Lambda^2} \left[\left(\frac{1}{2} \nabla^\lambda R + \nabla^\lambda L_m \right) \left(\frac{1}{2} \nabla_\lambda R + \nabla_\lambda L_m \right) g_{\mu\nu} - \left(\frac{1}{2} \nabla_\mu R + \nabla_\mu L_m \right) \left(\frac{1}{2} \nabla_\nu R + \nabla_\nu L_m \right) \right]. \quad (47)$$

In the case of weak gravitational fields and of small particle velocities, the exponential curvature-matter coupling induces an extra acceleration of massive test particles, which is proportional to the gradients of both the Ricci scalar and the matter Lagrangian, $\vec{a} = -(1/\Lambda)[\nabla(R/2) + \nabla L_m]$. In the case of pressureless dust, the extra force is proportional to the gradient of the matter density ρ only.

As one can see from Eqs. (47), the gravitational field equations of the exponential $f(R, L_m)$ gravity model contain a new background term, proportional to the metric tensor $g_{\mu\nu}$, which depends on both the constant term

Λ , and on the physical parameters of the matter. Thus, through the geometry–matter coupling, the exponential model introduces an effective, time dependent “cosmological constant”.

From Eqs. (6) and (7), we have

$$\Lambda_{eff}(R, L_m) = \Lambda - L_m(r), G_{eff}(R, L_m) = 1. \quad (48)$$

Then, for a Kasner-type metric, the field equations (37)–(40) describing the static, cylindrically symmetric, string in exponential $f(R, L_m)$ gravity take the form

$$\frac{(a+b+c-1)a}{r^2} = \rho(r) + \Lambda - L_m(r) - \frac{2R_0 - \Lambda r^2(2a+b+c)}{\Lambda^2 r^3} L'_m(r) + \frac{R_0 [\Lambda r^2(2a+b+c-3) - R_0]}{\Lambda^2 r^6} - \frac{L''_m(r)}{\Lambda} - \frac{L'^2_m}{\Lambda^2}, \quad (49)$$

$$\frac{a^2 + b^2 + c^2 - (a+b+c)}{r^2} = -p_r(r) + \Lambda - L_m(r) + \frac{a+b+c}{\Lambda r^4} [r^3 L'_m(r) - R_0], \quad (50)$$

$$\frac{(a+b+c-1)b}{r^2} = -p_\phi(r) + \Lambda - L_m(r) + \frac{[\Lambda r^2(a+c) - 2R_0]}{\Lambda^2 r^3} L'_m(r) + \frac{L'^2_m}{\Lambda^2} + \frac{R_0 [R_0 - \Lambda r^2(a+c-3)]}{\Lambda^2 r^6} + \frac{L''_m(r)}{\Lambda}, \quad (51)$$

$$\frac{(a+b+c-1)c}{r^2} = \rho(r) + \Lambda - L_m(r) + \frac{\Lambda r^2(a+b) - 2R_0}{\Lambda^2 r^3} L'_m(r) + \frac{R_0 [R_0 - \Lambda r^2(a+b-3)]}{\Lambda^2 r^6} + \frac{L''_m(r)}{\Lambda} + \frac{L'^2_m(r)}{\Lambda^2}, \quad (52)$$

where we have also used the explicit form of the Ricci scalar for the Kasner metric, given by Eq. (42). Equations (49) and (52) immediately give the following differential equation, which is satisfied by the matter Lagrangian;

$$\frac{L''_m(r)}{\Lambda} + \frac{L'^2_m(r)}{\Lambda^2} - \frac{4R_0 - \Lambda r^2(3a+2b+c)}{2\Lambda^2 r^3} L'_m(r) - \frac{R_0 [\Lambda r^2(3a+2b+c-6) - 2R_0]}{2\Lambda^2 r^6} + \frac{(a+b+c-1)(c-a)}{2r^2} = 0. \quad (53)$$

By introducing a new dependent variable $l_m(r)$, defined via $L'_m(r) = \Lambda l'_m(r)/l_m(r)$, $L_m(r) = \Lambda \ln l_m(r)$, Eq. (53) takes the form of a linear differential equation for $l_m(r)$, given by

$$l''_m(r) - \frac{4R_0 - \Lambda r^2(3a+2b+c)}{2\Lambda r^3} l'_m(r) - \left\{ \frac{R_0 [\Lambda r^2(3a+2b+c-6) - 2R_0]}{2\Lambda^2 r^6} - \frac{(a+b+c-1)(c-a)}{2r^2} \right\} l_m(r) = 0. \quad (54)$$

By introducing a new independent variable $\xi = 1/r$, we have $l'_m = (dl_m/d\xi)(d\xi/dr) = -\xi^2 dl_m/d\xi$, and $l''_m = 2\xi^3 (dl_m/d\xi) + \xi^4 (d^2 l_m/d\xi^2)$, respectively. Therefore Eq. (54) takes the form

$$\frac{d^2 l_m(\xi)}{d\xi^2} - \frac{(\beta-2)\xi^2 - s}{2\xi} \frac{dl_m(\xi)}{d\xi} -$$

$$\left(\mu - \frac{\beta}{2}\xi^2 - \frac{m}{\xi^2} \right) l_m(\xi) = 0, \quad (55)$$

where we have denoted $\beta = 4R_0/\Lambda$, $s = 3a+2b+c$, $\mu = \beta(s-6)/8$, and $m = (a+b+c-1)(c-a)/2$. Equation (55) has the general solution

$$l_m(\xi) = l_m^0 e^{m_1 \xi^2} \xi^{m_2} \left[c_1 U \left(u_1, u_2, -\frac{1}{4} \sqrt{(\beta-12)\beta+4\xi^2} \right) + c_2 L_{-u_1}^{u_2-1} \left(-\frac{1}{4} \sqrt{(\beta-12)\beta+4\xi^2} \right) \right], \quad (56)$$

where $U(a, b, z)$ is the confluent hypergeometric function, defined as $U(a, b, z) = [1/\Gamma(a)] \int_0^\infty e^{-zt} t^{a-1} (1+t)^{b-a-1} dt$, $L_n^a(x)$ is the generalized Laguerre polynomial, satisfying the differential equation $xy'' + (a+1-x)y' + ny = 0$, and where we have denoted

$$l_m^0 = 2^{\frac{1}{8}} (\sqrt{(s-2)^2 - 16m} + 4), \quad (57)$$

$$m_1 = \frac{1}{8} \left(\beta + \sqrt{(\beta-12)\beta+4} - 2 \right), \quad (58)$$

$$m_2 = \frac{1}{4} \left(\sqrt{(s-2)^2 - 16m} - s + 2 \right), \quad (59)$$

$$u_1 = \frac{(s+2)\beta - 2(s+4\mu+2) + \sqrt{(s-2)^2 - 16m} \sqrt{(\beta-12)\beta+4} + 4\sqrt{(\beta-12)\beta+4}}{8\sqrt{(\beta-12)\beta+4}}, \quad (60)$$

and

$$u_2 = \frac{1}{4} \left(\sqrt{(s-2)^2 - 16m} + 4 \right). \quad (61)$$

The parameters c_1 and c_2 are arbitrary integration constants. Hence, the thermodynamical variables of the Kasner string in the exponential type $f(R, L_m)$ gravity can

be expressed in terms of the function $l_m(r)$ as

$$\rho(r) = \Lambda [\ln l_m(r) - 1] + \frac{a+c}{2r} \frac{l'_m(r)}{l_m(r)} + \frac{(a+b+c-1)(3a-c)}{2r^2} - \frac{(a+c)R_0}{2\Lambda r^4}, \quad (62)$$

$$p_r(r) = \Lambda [1 - \ln l_m(r)] + \frac{(a+b+c) - (a^2+b^2+c^2)}{r^2} + \frac{a+b+c}{\Lambda r^4} \left[\Lambda r^3 \frac{l'_m(r)}{l_m(r)} - R_0 \right], \quad (63)$$

$$p_\phi(r) = \Lambda [1 - \ln l_m(r)] + \frac{c-a-2b}{2r} \frac{l'_m(r)}{l_m(r)} - \frac{(a+b+c-1)(c-2b-a)}{r^2} + \frac{(a+b-2c)R_0}{2\Lambda r^4}. \quad (64)$$

In the limit of small ξ , corresponding to large r , Eq. (56) can be approximated as $l_m(\xi) \propto \xi^{m_2}$, or, equivalently, $l_m(r) \propto r^{-m_2}$. Thus it follows that, at large radial distances, $L_m(r) \approx -m_2\Lambda \ln r$ and $l'_m(r)/l_m(r) \approx -m_2/r$. By using these approximations for $L_m(r)$ and $l_m(r)$ in Eqs. (62)–(64), we can easily determine the behavior of the energy density and thermodynamic pressures of the Kasner string in the exponential $f(R, L_m)$ gravity theory.

In the same order of approximation we obtain for the gravitational Lagrangian density and its derivative with respect to R the expressions

$$f(R, L_m) \approx \Lambda r^{-m_2} \exp\left(\frac{R_0}{2\Lambda r^2}\right),$$

$$f_R(R, L_m) \approx \frac{1}{2} r^{-m_2} \exp\left(\frac{R_0}{2\Lambda r^2}\right). \quad (65)$$

By using the results above we can now estimate the mass per unit length and angular deficit parameter of the string in the exponential model. We assume that the string core extends between $r = 0$ and $r = R_s$, and thus we avoid the singularity at the upper integration limits in the integrals. The string radius can be estimated from the equation $p_r(R_s) = 0$, which gives

$$\Lambda [1 + m_2 \ln R_s] + \frac{(a+b+c) - (a^2+b^2+c^2)}{R_s^2} - \frac{a+b+c}{\Lambda R_s^4} [\Lambda m_2 R_s^2 + R_0] = 0. \quad (66)$$

By neglecting the term $\Lambda [1 + m_2 \ln R_s]$ in Eq. (66), we obtain for the string radius the second-order algebraic equation

$$[(a+b+c) - (a^2+b^2+c^2)] R_s^2 + m_2(a+b+c) R_s^2 + (a+b+c) \frac{R_0}{\Lambda} = 0. \quad (67)$$

If the Kasner conditions hold at least approximately, so that $a+b+c \approx 1$ and $a^2+b^2+c^2 \approx 1$, the string radius can be approximated as $R_s \approx \sqrt{-R_0/m_2\Lambda}$.

Hence, for the mass and the angular deficit parameter, we obtain

$$M(r) = 2\pi\beta\chi(a, b, c)(kr)^{2a+b-1} + \frac{4\pi\beta}{k^2} \frac{(kr)^{2a+b+1}}{2a+b+1} \Lambda - 2\beta k^4 (kR_s)^{2a+b-5} \left[\frac{R_0 R_s^2 (2a+b+c-2m_2-3)}{\Lambda(2a+b-3)} + \frac{m_2 R_s^4 (2a+b+c-m_2-1)}{2a+b-1} - \frac{2\Lambda m_2 R_s^6}{(2a+b+1)^2} + \frac{2\Lambda m_2 R_s^6 \log(R_s)}{2a+b+1} - \frac{R_0^2}{\Lambda^2(2a+b-5)} \right] \quad (68)$$

and

$$W(r) = -2\pi\beta\theta(a, b, c)(kr)^{2a+b-1} - \frac{4\pi\beta}{k^2} \frac{(kr)^{2a+b+1}}{2a+b+1} \Lambda + 2\pi\beta k^4 (kR_s)^{2a+b-5} \left[\frac{R_0 R_s^2 (-a+b-c+2m_2+3)}{\Lambda(2a+b-3)} + \frac{m_2 R_s^4 (-a+b-c+m_2+1)}{2a+b-1} + \frac{2\Lambda m_2 R_s^6}{(2a+b+1)^2} - \frac{2\Lambda m_2 R_s^6 \log(R_s)}{2a+b+1} + \frac{R_0^2}{\Lambda^2(2a+b-5)} \right], \quad (69)$$

where

$$\chi(a, b, c) = (1-b)(a+b+c) + b - \frac{(a^2+b^2+c^2)}{(2a+b-1)}, \quad (70)$$

and

$$\theta(a, b, c) =$$

$$\frac{[(a+b+c-1)(a-b+c-1)+a^2+b^2+c^2-1]}{(2a+b-1)}. \quad (71)$$

In order to avoid any divergence at the origin, the parameters a and b must satisfy the condition $2a+b > 6$.

By assuming that the parameter Λ in the exponential model is the cosmological constant, we may assume for it a numerical value of the order of $\Lambda \approx 10^{-56} \text{ cm}^{-2}$. Then we can estimate the string radius as $R_s \approx (-R_0/m_2)^{1/2} \times 10^{28} \text{ cm}$. This relation imposes the constraint $R_0/m_2 < 0$. However, this value of the radius is obtained in the quasi-Kasner approximation, which implies that R_0 may have very small values, $R_0 = \epsilon \ll 1$, while m_2 still can have values of the order of unity. Therefore the radius of the string can be written as $R_s \approx \sqrt{\epsilon} \times 10^{28} \text{ cm}$. To obtain physically reasonable values of the radius, very small deviations from the exact Kasner regime are required. For example, with $\epsilon = 10^{-40}$, we can obtain more realistic string radius values of the order of $R_s \approx 10^8 \text{ cm}$. By taking into account the approximate expression of R_s in Eq. (68), we can estimate the mass per unit length of the string as

$$M \approx 2\beta \left(\frac{c_{light}^2}{\mathcal{G}} \right) (kR_s)^{2a+b-1} \\ \approx 2.7 \times 10^{28} \times \beta \times (kR_s)^{2a+b-1} \text{ g/cm}. \quad (72)$$

Besides the parameters a and b , the mass of the string depends on the other two parameters β and k of the Kasner metric. For $k \approx 1/R_s$, and $\beta = 10^{-4}$, we obtain for the mass per unit length of the string $M \approx 10^{24} \text{ g/cm}$. Therefore ultralong cosmic strings, with lengths of the order of $1 \text{ kpc} = 3 \times 10^{21} \text{ cm}$, could reach masses of order $M \approx 1.5 \times 10^{12} M_\odot$ in this model. Of course these estimates are strongly dependent on the values of β , k and the parameters of the Kasner metric, whose exact values can be obtained only by fitting with astrophysical data.

In the first-order approximation the angular deficit parameter of the string can be written in a form very similar to the Tolman mass equation,

$$W \approx -2\pi\beta\theta(a, b, c) \left(\frac{c_{light}^2}{\mathcal{G}} \right) (kR_s)^{2a+b-1} \text{ g/cm}. \quad (73)$$

The basic difference between the numerical values of M and W is determined by the functional form of $\theta(a, b, c)$, which, as one can see from Eq. (71), in the quasi-Kasner limit takes very small values, being exactly equal to zero in the standard Kasner case. So the presence of a nonzero negative angular deficit is determined by the deviation of the string interior metric from the exact Kasner form.

A. Observational implications

Gravitational lensing is one of the important physical effects that could, at least in principle, discriminate

between cosmic strings and other matter distributions. In the standard string scenario a vacuum string does not change the curvature of the space-time, but only its topology [68]. Therefore light rays are not deflected by a cosmic string. However, because of the specific conical structure of the space-time metric, if two light rays pass on the different sides of the string, they may converge later at the same point of observation [72]. If a cosmic string lies between us and a distant source, we see two images of the source separated by an angle $\delta\theta = 8\pi\mu_s \sin\alpha D_{LS}/D_{OS}$, where μ_s is the string linear density, α is the angle between the string and the observer - source direction, and D_{LS} and D_{OS} indicate the distance between the lens and the source, and the observer and source, respectively. Hence in the case of the standard conical string, the two images formed due to the string presence are identical to the original source, without presenting any amplification or distortion [72]. By contrast, the gravitational lensing by gas filaments shows a very different pattern, with one or three elongated images [72].

The gravitational lensing patterns of the $f(R, L_m)$ cosmic strings are also fundamentally different from the lensing of the conical strings, being closer to the lensing patterns produced by long gas filaments. The metric outside the $f(R, L_m)$ cosmic string is of Kasner-type, and therefore the propagation of light is strongly affected by the space-time geometry. By assuming, in the first approximation, that the interior density profile of the $f(R, L_m)$ cosmic string can be approximated by a Gaussian distribution with width σ_0 , for the lensing angle of a light ray passing at a distance D from the string central axis, consisting of a massive cylindrical matter distribution, we obtain the expression $\alpha \approx \left(\mathcal{G}M/c_{light}^2 \right) \text{Erf}(D/\sigma_0\sqrt{2})$, where $\text{Erf}(z)$ is the error function [72]. This equation is valid under the assumption that the length of the string is much larger than D . Hence, with the use of Eq. (72), we obtain for the light deflection angle the expression

$$\alpha \approx 2\beta (kR_s)^{2a+b-1} \text{Erf} \left(\frac{D}{\sigma_0\sqrt{2}} \right). \quad (74)$$

As one can see from the above equation, the lensing angle therefore depends on the intrinsic parameters k and β of the metric, as well as on the Kasner coefficients a and b . There is also a strong dependence on the string radius R_s , as well as on the matter density profile inside the string core. Due to the presence of a larger number of parameters in the deflection angle formula, cosmic strings in exponential $f(R, L_m)$ gravity could potentially provide a better fit to the observational data as compared to the standard string model. Also these massive string solutions predict a much richer lensing behavior, as compared to the simple conical string model, leading to the possibility of a much wider range of deflection angles.

Different aspects of lensing by cosmic strings have been analyzed in detail in the literature [73, 74]. A potential lensing effect, which may provide some evidence for the existence of cosmic strings, comes from the

Capodimonte–Sternberg lens candidate (CSL-1) [73]. However, detailed comparisons of observations with existing string lensing models have proved, “beyond any doubts” [73], that the CSL-1 images are simply a rather peculiar pair of interacting elliptical galaxies. Nonetheless, a future improvement in the precision of the observations could lead to the potential identification of $f(R, L_m)$ strings, in the sense that lensing from such strings may fit CSL-1 data for appropriate choices of the model parameters.

V. STRING SOLUTIONS IN $f(R, L_m)$ GRAVITY WITH SELF-CONSISTENT MATTER LAGRANGIAN

In the present section we consider some explicit solutions of the field equations (21)–(24), without imposing *a priori* the functional form of $f(R, L_m)$.

From Eqs. (37) and (40) it immediately follows that the function f_R satisfies the second-order linear differential equation

$$\frac{d^2}{dr^2} f_R + \frac{3a+2b+c}{r} \frac{d}{dr} f_R = \frac{(a+b+c-1)(c-a)}{r^2} f_R, \quad (75)$$

which has the general solution

$$f_R(r) = C_1'' r^{\alpha_1} + C_2'' r^{\alpha_2}, \quad (76)$$

where C_1'' and C_2'' are arbitrary constants of integration and α_1 and α_2 are solutions of the algebraic equation

$$\alpha^2 + (3a+2b+c-1)\alpha - (a+b+c-1)(c-a) = 0, \quad (77)$$

given by

$$\alpha_{1,2} = \frac{1-3a-2b-c}{2} \pm \frac{\sqrt{(3a+2b+c-1)^2 + 4(a+b+c-1)(c-a)}}{2}. \quad (78)$$

Using $R = R_0/r^2$, where R_0 is given by Eq. (42), we can rewrite Eq. (76) as a function of R as

$$f_R(R, L_m) = C_1' R^{-\alpha_1/2} + C_2' R^{-\alpha_2/2}, \quad (79)$$

where $C_1' = R_0^{\alpha_1/2} C_1''$ and $C_2' = R_0^{\alpha_2/2} C_2''$ giving, immediately,

$$f(R, L_m) = C_1 R^{n_1} + C_2 R^{n_2} + g(L_m), \quad (80)$$

where $n_1 = 1 - \alpha_1/2$, $n_2 = 1 - \alpha_2/2$, $C_1 = C_1'/(1 - \alpha_1/2)$, $C_2 = C_2'/(1 - \alpha_2/2)$, and $g(L_m)$ is an arbitrary integration function of the matter Lagrangian L_m . This encompasses the solution obtained in [32] (which corresponds to the condition $f(R=0) = 0$, giving $f(R)$ as a superposition of powers R^k , $k \geq 1$), when both sets of Kasner conditions, Eq. (36), are satisfied and $g(L_m) = 0$.

However, it is clearly more general and holds for $R \neq 0$, $g(L_m) \neq 0$.

The components of the string energy-momentum tensor can then be obtained as

$$\rho = \frac{1}{G_{eff}(R, L_m)} \left[\frac{(a+b+c-1)c}{r^2} - \Lambda_{eff}(R, L_m) - \frac{a+b}{r} \frac{d}{dr} \ln f_R(R, L_m) \right], \quad (81)$$

$$p_r = \frac{1}{G_{eff}(R, L_m)} \left[\frac{(a+b+c) - (a^2+b^2+c^2)}{r^2} + \Lambda_{eff}(R, L_m) + \frac{a+b+c}{r} \frac{d}{dr} \ln f_R(R, L_m) \right] \quad (82)$$

$$p_\phi = \frac{1}{G_{eff}(R, L_m)} \left[\frac{(a+b+c-1)(c-a-b)}{r^2} + \Lambda_{eff}(R, L_m) - \frac{2(a+b)}{r} \frac{d}{dr} \ln f_R(R, L_m) \right] \quad (83)$$

$$p_z = -\rho. \quad (84)$$

By requiring that the generalized gravitational Lagrangian density must have the standard general relativistic limit, that is, by requiring it to reduce to the standard Einstein–Hilbert Lagrangian for an appropriate choice of the free parameters of the model, we can impose the conditions

$$n_1 = 1, C_1 = \frac{1}{2}, \quad (85)$$

which, according to Eq. (80), require $\alpha_1 = 0$. This criterion is satisfied if the coefficients a , b , c satisfy the condition

$$(a+b+c-1)(c-a) = 0. \quad (86)$$

Then, for α_2 we obtain $\alpha_2 = 1 - 3a - 2b - c$, giving the expression

$$n_2 = n = \frac{(1+3a+2b+c)}{2} \quad (87)$$

for $n_2 = 1 - \alpha_2/2$, where, for simplicity, we rename n_2 as n from now on.

Therefore the self-consistent gravitational Lagrangian for a string in $f(R, L_m)$ gravity takes the form

$$f(R, L_m) = \frac{R}{2} + \frac{C_2}{n} R^n + g(L_m), \quad (88)$$

which reduces to the Einstein–Hilbert Lagrangian when $C_2 = 0$ and $g(L_m) \rightarrow L_m$.

A. The energy conservation equation

In order to completely solve the problem of the determination of the generalized gravitational action for a string-like configuration, one must determine the dependence of the function $f(R, L_m)$ on the matter Lagrangian L_m . This can be achieved by substituting the expressions of the energy-momentum tensor, given by Eqs. (81)–(84), into the energy conservation equation, Eq. (41). The resulting equation completely determines the functional form of f , as well as the dependence of the matter Lagrangian on the

coordinate r . Taking into account the conditions given by Eq. (86), we have to consider two cases independently.

1. The case $a + b + c = 1$, $c \neq a$

After substituting the expressions for the string energy density and pressures into the conservation equation, Eq. (41), and by taking into account the conditions $a + b + c = 1$, $c \neq a$, we obtain the following differential equation, giving the dependence of the function $g(L_m(r))$ on the matter Lagrangian and on the radial coordinate r ,

$$R_0 \left\{ -2 \left[b^2 + b(c-1) + (c-1)c \right] + r^3 (-L_m(r)) L'_m(r) g''(L_m(r)) + R_0 \right\} - 2C_2 r^2 \left(\frac{R_0}{r^2} \right)^n \left[b^2(4n-2) + 2b(c-1)(5n-4) + c^2(4n-2) + c(4-6n) + 4(n-1)n - R_0 \right] = 0. \quad (89)$$

2. The case $a + b + c \neq 1$, $c = a$

The second case in which the condition given by Eq. (86) is satisfied is when the coefficients a , b , c in the Kasner metric satisfy the conditions $a + b + c \neq 1$ and $c = a$. For these values of the Kasner coefficients the energy conservation equation Eq. (41) takes the form

$$R_0 \left[-6a^2 - 4a(b-1) - 2(b-1)b + r^3 (-L_m(r)) L'_m(r) g''(L_m(r)) + R_0 \right] - 2C_2 r^2 \left(\frac{R_0}{r^2} \right)^n \left[4n^2(2a+b) - 2(2b+3)n(2a+b) + 6(a+b)^2 - R_0 \right] = 0. \quad (90)$$

VI. KASNER-TYPE STRING SOLUTIONS IN $f(R, L_m)$ GRAVITY WITH SELF-CONSISTENT GRAVITATIONAL LAGRANGIAN

In the present section we present some explicit, exact, string-type solutions of the gravitational field equations in $f(R, L_m)$ gravity, for which the source term satisfies the condition $T_t^t = T_z^z$ and the metric is assumed to be of the Kasner form. The solutions are obtained by imposing specific conditions on the coefficients a , b , c , or by assuming a specific functional form for the integration function $g(L_m(r))$.

A. Solutions satisfying both Kasner conditions

We consider first string-type solutions that satisfy both the Kasner conditions given in Eqs. (36). In this case

$R_t^t = R_r^r = R_\phi^\phi = R_z^z = 0$, and $R = 0$, respectively. The conservation equation Eq. (89) reduces to the condition $g''(L_m) = 0$, giving $g(L_m) = \text{constant} + L_m$. Then, by taking into account the fact that $f_R(R, L_m) = 0$, from the gravitational field equations (3), we obtain $T_{\mu\nu} = 0$, implying that $L_m = 0$. We therefore obtain again, in the framework of $f(R, L_m)$ gravity theory, the Kasner vacuum solution of standard general relativity.

B. Kasner-type string solutions with $g(L_m) = g_0 L_m^q(r)$

As a second example of a string solution in $f(R, L_m)$ gravity we consider the simple case in which $g(L_m)$ has a power law dependence on L_m , so that $g(L_m) = g_0 L_m^q(r)$, where g_0 and q are constants.

With this choice of the matter term the conservation equation (89) and Eq. (90) give the r -dependence of the

matter Lagrangian, allowing us to obtain, explicitly, the full solution of the gravitational field equations for the Kasner string. In the following we consider the cases $a + b + c = 1$, $c \neq a$ and $a + b + c \neq 1$, $c = a$ separately.

1. The case $a + b + c = 1$, $c \neq a$

If the values of the constants a , b , c satisfy the constraints $a + b + c = 1$, $c \neq a$, then from Eq. (89), for $g(L_m) = g_0 L_m^q(r)$, we obtain first the differential equation satisfied by the matter Lagrangian as

$$\begin{aligned} & -2C_2 r^2 L_m(r) \left(\frac{R_0}{r^2} \right)^n [b^2(4n-2) + 2b(c-1)(5n-4) + c^2(4n-2) + c(4-6n) + 4(n-1)n - R_0] \\ & + R_0 L_m(r) (R_0 - 2(b^2 + b(c-1) + (c-1)c)) - g_0(q-1)qr^3 R_0 L_m^q(r) L'_m(r) = 0, \end{aligned} \quad (91)$$

with the general solution given by

$$L_m(r) = 2^{-1/q} \left\{ \frac{r^{-2(n+1)} (2C_1 A r^2 - n R_0 r^{2n} [-2(b^2 + b(c-1) + c^2) - 2c_1 g_0(q-1)qr^2 + 2c + R_0])}{g_0 n(q-1)R_0} \right\}^{1/q}, \quad (92)$$

where c_1 is an arbitrary constant of integration, and we have denoted

$$A = R_0^n \left[b^2(4n-2) + 2b(c-1)(5n-4) + c^2(4n-2) + c(4-6n) + 4(n-1)n - R_0 \right]. \quad (93)$$

Hence we obtain the gravitational Lagrangian density explicitly as

$$f(R, L_m) = \frac{R}{2} + \frac{2}{2+2a+b} C_2 R^{(2+2a+b)/2} + g_0 L_m^q. \quad (94)$$

Imposing $2a + b = 2$ or, equivalently, $c = a - 1$, and setting $C_2 = 1/\Lambda$, we obtain the model $f(R, L_m) =$

$\frac{1}{2}(R + R^2/\Lambda^2) + g_0 L_m^q$, whose $f(R)$ version with matter minimally coupled to geometry is commonly studied [32]. Hence, static, string-type solutions *do* exist in the $f(R, L_m)$ cosmology, but their self-consistent description requires $R \propto r^{-2} \neq \text{constant}$ within the string core.

C. The case $a + b + c \neq 1$, $c = a$

If the parameters of the Kasner metric satisfy the conditions $a + b + c \neq 1$, $c = a$, then for $g(L_m) = g_0 L_m^q$, from the conservation Eq. (90), it follows that the matter Lagrangian satisfies the differential equation

$$\begin{aligned} & R_0 [-6a^2 - 4a(b-1) - 2(b-1)b + R_0] - 2C_2 r^2 \left(\frac{R_0}{r^2} \right)^n [4n^2(2a+b) - 2(2b+3)n(2a+b) + 6(a+b)^2 - R_0] \\ & - g_0(q-1)qr^3 R_0 L_m^{q-1}(r) L'_m(r) = 0, \end{aligned} \quad (95)$$

with the general solution given by

$$L_m(r) = 2^{-1/q} \left\{ \frac{2C_2 B r^2 - n R_0 r^{2n} [-6a^2 - 4a(b-1) - 2(b-1)b - 2c_2 g_0(q-1)qr^2 + R_0]}{g_0 n(q-1)R_0 r^{2(n+1)}} \right\}^{1/q}, \quad (96)$$

where c_2 is an arbitrary integration constant, and we have denoted

$$B = R_0^n [4n^2(2a+b) - 2(2b+3)n(2a+b) + 6(a+b)^2 - R_0]. \quad (97)$$

The gravitational Lagrangian of this model takes the form

$$f(R, L_m) = \frac{R}{2} + \frac{2}{1+2b+4c} C_2 R^{(1+2b+4c)/2} + g_0 L_m^q. \quad (98)$$

Imposing $a = c = \frac{1}{4}(3 - 2b)$, $C_2 = 1/\Lambda$, we again obtain the model $f(R, L_m) = \frac{1}{2}(R + R^2/\Lambda^2) + g_0 L_m^q$, whose $f(R)$ limit, corresponding to $q = 1$ and $g_0 = 1$, has been extensively studied. Therefore, in general, it is clear that in $f(R, L_m)$ gravity theories two *different* families of self-consistent Kasner-type string solutions exist.

$a + b + c \approx 1$, and $a^2 + b^2 + c^2 \approx 1$, but $R_0 = a^2 + b^2 + c^2 + (a + b + c)(a + b + c - 2) \neq 0$, implying that the Ricci scalar R is again nonzero inside the string. For the function $g(L_m)$ we adopt again a power law form, with $g(L_m) = g_0 L_m^q(r)$. Under these assumptions the energy-conservation equation (41) becomes

D. Quasi-Kasner solutions of the field equations

Finally, we consider the case in which both Kasner conditions, given by Eqs. (36) hold *approximately*, so that

$$2C_2 r^2 \left(\frac{R_0}{r^2} \right)^n [R_0 - 2(n-1)(3b(c-1) - c + 2n)] + R_0^2 - g_0(q-1)qr^3 R_0 L_m(r)^{q-1} L'_m(r) = 0, \quad (99)$$

with the general solution for L_m given by

$$L_m(r) = 2^{-1/q} \left\{ \frac{2C_2 r^2 R_0^n [2(n-1)(3b(c-1) - c + 2n) - R_0] - nR_0 r^{2n} [R_0 - 2c_3 g_0(q-1)qr^2]}{g_0 n(q-1)R_0 r^{2(n+1)}} \right\}^{1/q}, \quad (100)$$

where c_3 is an arbitrary constant of integration.

The thermodynamic parameters of the quasi-Kasner string are obtained as

$$\rho(r) = \frac{\{-2C_2 r^2 (R_0/r^2)^n [4(-1+c)(-1+n)n + R_0] + nR_0 [-R_0 + 2g_0(-1+q)r^2 L_m^q(r)]\} L_m^{1-q}(r)}{4g_0 n q r^2 R_0}, \quad (101)$$

$$p_r(r) = \frac{L_m^{1-q}(r) \left\{ nR_0 [R_0 - 2g_0(q-1)r^2 L_m^q(r)] - 2C_2 r^2 [4(n-1)n - R_0] \left(\frac{R_0}{r^2} \right)^n \right\}}{4g_0 n q r^2 R_0}, \quad (102)$$

$$p_\phi(r) = \frac{L_m^{1-q}(r) \left\{ nR_0 [R_0 - 2g_0(q-1)r^2 L_m^q(r)] - 2C_2 r^2 [8(c-1)(n-1)n - R_0] \left(\frac{R_0}{r^2} \right)^n \right\}}{4g_0 n q r^2 R_0}, \quad (103)$$

$$p_z(r) = -\rho(r). \quad (104)$$

With the use of Eqs. (6) and (7) we obtain for G_{eff} and Λ_{eff} the general expressions

$$G_{eff} = \frac{g'(L_m)}{1 + 2C_2 R_0^{n-1}/r^{2(n-1)}} = \frac{g_0 q L_m^{q-1}(r)}{1 + 2C_2 R_0^{n-1}/r^{2(n-1)}}, \quad (105)$$

$$\Lambda_{eff} = \frac{1}{1 + 2C_2 R_0^{n-1}/r^{2(n-1)}} \times \left[\frac{R}{2} + \frac{C_2 R_0^n}{n r^{2n}} + g(L_m) - g'(L_m) L_m \right]$$

$$= \frac{1}{1 + 2C_2 R_0^{n-1}/r^{2(n-1)}} \left[\frac{R}{2} + \frac{C_2}{n} \frac{R_0^n}{r^{2n}} + g_0(1-q)L_m^q(r) \right], \quad (106)$$

respectively.

Again, we may recover the common model with R and R^2 terms for an appropriate choice of the Kasner parameters, but the important point to note is that, for all cases in which self-consistent Kasner-type string solutions exist, other more general models are also possible. In particular, there exist two distinct families of nonvacuum Kasner-type solutions for which the Kasner conditions are *not* satisfied inside the string core, and specific relations between the parameters a , b and c must be imposed in order to satisfy the correspondence principle, embodied in Eq. (86).

However, an additional family of string-type solutions also exists in which the Kasner conditions are satisfied *approximately*, but no further conditions relating a , b and c need to be imposed to ensure the existence of the general relativistic limit. Setting $C_2^{-1} = R_0 = 2\Lambda$ and $n = 2$ in Eq. (88) is one way to obtain an effective cosmological constant (to leading order for large r) at the level of the field equations, which is naively consistent with the results of the previous work in which Linet–Tian type solutions (corresponding to the exterior of a cosmic string embedded in a space-time with $\Lambda \neq 0$ [50]) were found to exist in $f(R)$ gravity [32, 33].

Thus, it may be hoped that future studies, in which the local EOM for field-theoretic variables in the matter Lagrangian are explicitly solved in conjunction with the gravitational field equations, recover the solutions obtained in [32, 33] as $T_t^t(r) = T_z^z(r) \rightarrow 0$ for $r \rightarrow \infty$; that is, as the string energy density tends asymptotically to zero. For small r , one would also hope to be able to relate the parameters in the Kasner metric to the fundamental

field theory parameters (e.g. the symmetry breaking energy scale, η , scalar and vector coupling constants, λ and e , and topological winding number, $|n|$ in the Abelian-Higgs model), directly, via expressions such as Eq. (88).

E. The mass and angular deficit of the string

In the following we limit our analysis to the quasi-Kasner case, that is, we assume that $a + b + c \approx 1$, $a^2 + b^2 + c^2 \approx 1$, but $R_0 \neq 0$. The radius R_s of the string can be obtained from the condition $p_r(R_s) = 0$, and it is obtained as a solution of the equation

$$nR_0 [R_0 - 2g_0(q-1)r^2 L_m^q(R_s)] - 2C_2 R_s^2 [4(n-1)n - R_0] \left(\frac{R_0}{R_s^2} \right)^n = 0. \quad (107)$$

In the string Lagrangian given by Eq. (100), inside the bracket we have a sum of three terms, the first varying as a function of r as $1/r^{2+2a+b}$, the second as $1/r^2$, and a third constant term. Therefore in the large distance limit the dominant term is $1/r^2$, and we can adopt for the string Lagrangian the approximate expression

$$L_m(r) \approx \left[\frac{R_0}{n(1-q)} \frac{1}{r^2} \right]^{1/q}. \quad (108)$$

For the radius of the string we then obtain the expression

$$R_s \approx \left\{ \frac{2[4n(n-1) - R_0]}{nR_0^2(1 + 2g_0/n)} \right\}^{1/2(n-1)} C_2^{1/2(n-1)}. \quad (109)$$

For this value of the radius both the energy density and the tangential pressure p_ϕ have nonzero values at the string surface.

The Tolman mass per unit length of the string is

$$M(R_s) \approx \frac{\pi\beta(1-q)}{g_0 R_0^2 k^{2(1-1/q)}} \left[\frac{R_0}{n(1-q)} \right]^{\frac{1}{q}} (kR_s)^{3-b-2c-2/q} \left\{ \frac{2C_2 R_0^n R_s^{2(1-n)} [2(2c-1)(n-1)n - R_0]}{q(b+2c+2n-5) + 2} - \frac{R_0^2(2g_0+n)}{q(b+2c-3) + 2} \right\}, \quad (110)$$

where, in order to avoid the singularity in the origin, we have to impose the conditions $b + 2(c + 1/q) < 3$,

and $b + 2(c + n + 1/q) < 5$, respectively. In the same approximation for the W parameter, which determines the angular deficit, we obtain

$$W(R_s) = \frac{\pi\beta(q-1)}{g_0 R_0^2 k^{2(1-1/q)}} \left[\frac{R_0}{n(1-q)} \right]^{\frac{1}{q}} (kR_s)^{3-b-2c-2/q} \left\{ \frac{2C_2 R_0^n R_s^{2(1-n)} [2(4c-5)(n-1)n + R_0]}{q(b+2c+2n-5) + 2} + \frac{R_0^2(2g_0+n)}{q(b+2c-3) + 2} \right\}, \quad (111)$$

where we again impose the same conditions in order to avoid the singularity at $r = 0$.

By assuming that $C_2 = C_0/\Lambda$, where Λ is the cosmological constant with values of the order of $\Lambda = 10^{-56} \text{ cm}^{-2}$, the radius of the string can be estimated as

$$R_s \approx \left\{ 2 \frac{[4(n-1)n - R_0]}{nR_0^2(1 + 2g_0/n)} \right\}^{1/2(n-1)} \times 10^{28/(n-1)} \text{ cm.} \quad (112)$$

The string radius thus rapidly decreases with increasing n , which determines the R -dependence of the gravitational action. In addition, very small values of $4n(n-1) - R_0$ can further reduce the string radius. Alternatively, since $n = 1 + a + b/2$, very small values of a and b can significantly increase the radius of the string. For $a + b/2 = 4$, we obtain values of the string radius of order $R_s \approx 10^7 \text{ cm}$. In the leading order of approximation the mass-radius relation of the quasi-Kasner string is given as

$$M(R_s) \propto (kR_s)^{5-b-2c-2/q-2n}, \quad (113)$$

or

$$M(R_s) \approx 1.35 \times 10^{28} (kR_s)^{5-b-2c-2/q-2n} \text{ g/cm.} \quad (114)$$

The mass of the string also depends on q , giving the matter Lagrangian dependence in the gravitational action. On the other hand, the mass of the string depends on the coefficient R_0 , which, due to the impositions of the approximate Kasner conditions, may have a very small value. Hence, depending on the adopted values of the model parameters, a very wide range of radii and masses can be generated for cosmic strings in $f(R, L_m)$ gravity. Similar relations can be obtained for the angular deficit W , $W \propto (kR_s)^{5-b-2c-2/q-2n}$, with the sign of W depending on the adopted numerical values of the string metric parameters. Under the same assumption of an approximate Gaussian matter density profile, the relation $\alpha \approx (GM/c_{\text{light}}^2) \text{Erf}(D/\sigma_0\sqrt{2})$ leads to a great variety of lensing behaviors, due to the strong dependence of the string mass on the model parameters.

VII. DISCUSSIONS AND FINAL REMARKS

In this paper we have considered static, cylindrically symmetric solutions of the gravitational field equations for a string-like distribution of matter in the $f(R, L_m)$ modified gravity theory, in which the gravitational Lagrangian is given by an arbitrary function of the Ricci scalar and matter Lagrangian. Having determined the gravitational field equations for a general cylindrically symmetric metric, with arbitrary dependence of the metric tensor components on the radial distance, r , we have restricted our subsequent analysis to a specific case by adopting the Kasner metric for the string interior, in

which the components of the metric tensor are proportional to powers of r .

Two distinct models have been investigated. In the first, the gravitational Lagrangian was fixed in the form of an exponential function, $f(R, L_m) = (1/\Lambda) \exp(R/2\Lambda + L_m/\Lambda)$, which obviously reduces to the Einstein–Hilbert Lagrangian in the limit $R/2 + L_m \ll \Lambda$. In the second, $f(R, L_m)$ was self-consistently determined from the field equations using the string condition, $T_t^t = T_z^z$, or equivalently, $\rho = -p_z$, where ρ is the mass per unit volume of the string and p_z is the thermodynamic pressure, together with the correspondence principle, which requires the Einstein–Hilbert limit to exist for appropriate choices of the free model parameters.

In the first case, from the field equations one can obtain a second-order linear differential equation for the matter Lagrangian, whose solution can be expressed in terms of the hypergeometric and Laguerre functions. The knowledge of the matter Lagrangian fully determines the solution of the field equations, allowing the determination of the energy density and anisotropic pressures inside the string. Crucially, in this case, the Kasner parameters are required to satisfy the relation $2a + b > 6$, in order to avoid divergences in the Tolman mass density (i.e. the relativistic mass per unit length of the string), M , and W parameter, which determines the angular deficit [59], as $r \rightarrow 0$.

In the second case, by imposing the string conditions for the field equations in the Kasner metric, we obtain a second-order differential equation for $f(R, L_m)$ which allows the determination of the gravitational Lagrangian density. In this case $f(R, L_m)$ has an additive structure, being the sum of a Ricci scalar-dependent function, and of a matter Lagrangian-dependent function $g(L_m(r))$, which is an arbitrary function of integration. The dependence on the Ricci scalar can be simplified with the use of the correspondence principle, by requiring the standard general relativistic limit of the modified gravity action (i.e. the Einstein–Hilbert action) to exist for appropriate choices of the arbitrary parameters and functions of the model. In order to determine the matter Lagrangian dependence one must use the conservation equation of the theory, which gives a strongly nonlinear differential equation for $g(L_m(r))$. In order to obtain some explicit solutions of the field equations we have assumed the simple case in which $g(L_m(r)) = g_0 L_m^q(r)$, $q \neq 1$. This choice allows the immediate determination of the matter Lagrangian from the conservation equation, and of the general solution of the field equations, for a gravitational model described by a Lagrangian density of the form $f(R, L_m) = R/2 + (C_2/n) R^n + g_0 L_m^q$, $q \neq 1$, where $n = n(a, b, c)$ depends on the parameters a, b, c of the Kasner metric. In this case, the function $n(a, b, c)$ may take two different forms, corresponding to two distinct sets of conditions imposed on the Kasner parameters, a, b, c , in order to satisfy the correspondence principle. Thus, depending on the choice of conditions for a, b and

c , two distinct string-type models can be obtained. An additional family of solutions, which we refer to as quasi-Kasner solutions, also exists, in which both sets of Kasner conditions (i.e. those defining both the Kasner sphere and the Kasner plane [48]) are approximately satisfied, but for which the Ricci scalar is nonzero inside the string core.

Importantly, for *all* sets of self-consistent Kasner-type string solutions obtained in our study, the Ricci scalar is of the form $R = R_0(a, b, c)/r^2$, where $R_0(a, b, c) = 0$ when both sets of Kasner conditions are satisfied. Thus we recover, in the context of the $f(R, L_m)$ theory of gravity, the unique $R = 0$ vacuum solution obtained in general relativity [48, 49] and, previously, in $f(R)$ gravity [32, 33]. However, the existence of vacuum solutions in the $f(R)$ theory for which $R = \text{const.} \neq 0$, also found in [32, 33], and which correspond to Linet–Tian (LT) type solutions [50] under the identification of $R = \text{const.}$ with the cosmological constant term in the gravitational field equations, imply that the Kasner metric fails to capture LT-type solutions in both $f(R)$ and, more generally, $f(R, L_m)$ gravity.

The physical interpretation of this is that, although cylindrically symmetric “Kasner-type” solutions, for which $R = 4\Lambda = \text{constant}$ may be identified with the cosmological constant, exist in general relativity [51], no such solutions exist for the Kasner metric in $f(R)$ and $f(R, L_m)$ gravity. Hence, since the (non-Kasner) solutions exhibiting cylindrical symmetry so far obtained in $f(R)$ gravity correspond to vacuum solutions for which $R = \text{constant} \neq 0$ and, since the unique $R = 0$ vacuum

solution obtained in [32, 33] is equally well expressed in terms of the Kasner metric, as shown in the present study, we may conclude that non-Kasner metrics are required to capture *all* additional static, cylindrically symmetric, vacuum solutions (for which $R \neq 0$) in both the $f(R)$ and the $f(R, L_m)$ modified gravity theories.

Observationally, these results are interesting since they provide a much richer gravitational lensing pattern for strings in $f(R, L_m)$ gravity, as compared with strings in general relativity. In particular, both the potential range of string widths and the range of lensing angles is far greater than in the standard scenario, and additional distortion of lensing images occurs due to the space-time curvature, even for vacuum strings. Since these also depend, sensitively, on the values of many free model parameters, it may even be hoped that future high-precision observations of anomalous lensing candidates, such as CSL-1, may yield data compatible with $f(R, L_m)$ string models.

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Appendix A: The conservation equation in $f(R, L_m)$ gravity

In this appendix we give an explicit proof of the “non-conservation” equation (14) in $f(R, L_m)$ gravity. By taking the covariant divergence of Eq. (3) we first obtain

$$\begin{aligned} & \nabla^\mu f_R R_{\mu\nu} + f_R \nabla^\mu R_{\mu\nu} + (\nabla_\nu \square - \square \nabla_\nu) f_R \\ & - \frac{1}{2} \left[f_R \nabla^\mu R + f_{L_m} \nabla^\mu L_m - \nabla^\mu f_{L_m} L_m - f_{L_m} \nabla^\mu L_m \right] g_{\mu\nu} \\ & = \frac{1}{2} \nabla^\mu f_{L_m} T_{\mu\nu} + \frac{1}{2} f_{L_m} \nabla^\mu T_{\mu\nu}. \end{aligned} \quad (A1)$$

With the help of Eq. (13) we then have

$$(\nabla_\nu \square - \square \nabla_\nu) f_R = -R_{\mu\nu} \nabla^\mu f_R, \quad (A2)$$

while from purely geometric considerations

$$\begin{aligned} & f_R \nabla^\mu R_{\mu\nu} - \frac{1}{2} f_R \nabla^\mu R g_{\mu\nu} \\ & = f_R \nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \equiv 0. \end{aligned} \quad (A3)$$

Therefore, Eq. (A1) reduces to

$$\nabla^\mu T_{\mu\nu} = \frac{1}{f_{L_m}} \nabla^\mu f_{L_m} (L_m g_{\mu\nu} - T_{\mu\nu}), \quad (A4)$$

or, equivalently,

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu \ln f_{L_m} (L_m g_{\mu\nu} - T_{\mu\nu}). \quad (A5)$$

Appendix B: Derivation of the Kasner metric in standard general relativity

In standard general relativity the vacuum gravitational field equations satisfy the conditions $R^\nu_\nu = 0$ and $R = 0$, respectively. For a general static cylindrically symmetric metric of the form (16), the components of the Ricci tensor are given by Eqs. (19). The equations $R^t_t = R^\phi_\phi = R^z_z = 0$ can be immediately integrated to give

$$LKN' = C_1, NKL' = C_2, NLK' = C_3, \quad (B1)$$

where C_1, C_2, C_3 are arbitrary constants of integration. From the above equations we obtain

$$\frac{N'}{N} = \alpha_1 \frac{L'}{L}, \frac{N'}{N} = \beta_1 \frac{K'}{K}, \frac{K'}{K} = \frac{1}{\gamma_1} \frac{L'}{L}, \quad (B2)$$

where $\alpha_1 = C_1/C_3$, $\beta_1 = C_1/C_3$, and $\gamma_1 = C_2/C_3$. The condition $R^r_r = 0$ then gives the following equation for $L(r)$,

$$(\alpha_1 \gamma_1 + \gamma_1 + 1) L(r) L''(r) - (\alpha_1 \gamma_1 + 1) L'^2 = 0, \quad (B3)$$

which has the general solution

$$L(r) = c_2 (\gamma_1 r)^{1+\alpha_1+1/\gamma_1} = c_2 (\gamma_1 r)^b = \beta (kr)^{b-1} r, \quad (B4)$$

where c_2 is an arbitrary integration constant and where we have taken, without any loss of generality, an arbitrary integration constant to be zero. We have also denoted $b = 1 + \alpha_1 + 1/\gamma_1$. In the last equality in Eq. (B4) we have rescaled the arbitrary integration constants. By denoting $a = \alpha_1 b$ and $c = b/\gamma_1$, we obtain

$$N(r) = N_0 (\gamma_1 r)^a, K(r) = K_0 (\gamma_1 r)^c, \quad (\text{B5})$$

where N_0 and K_0 are also arbitrary integration constants. Then, by using the above expressions for the metric tensor components, Eqs. (B1) impose the restriction $a + b + c = 1$ on the constants a, b, c . Finally, the condition $R = R_0/r^2 = 0$ requires $R_0 = 0$ for the vacuum solution and with the use of the explicit form of R_0 , given by Eq. (42), it follows that in order for the Kasner metric to represent a static general relativistic vacuum solution of the gravitational field equations, the coefficients a, b, c must also satisfy the second Kasner condition $a^2 + b^2 + c^2 = 1$.